Brane World Models Without Cosmological Expansive And Darboux Transformations.

A.V. Yurov
Theoretical Physics Department,
Kaliningrad State University, Russia, yurov@freemail.ru

Abstract

We consider 5-D gravity plus a bulk scalar field and with parallel 3-branes in the case when Hubble root is zero. To construct exact solutions we use the Darboux transformations. To do this we reduce the system of equations which describes 5-D gravity and bulk scalar field to the Schrödinger equation. The jump conditions at the branes lead to jump potential in the Schrödinger equation so we need to modify usual Darboux-Crum formulas in this case. Using this transformation we have constructed new exact solutions of brane equations which is some generalization of the Rundall-Sundrum solution.

1 Introduction.

Recently, E.E. Flanagan, S.-H.Tye and I. Wasserman [1] considered 5-D gravity plus a bulk scalar field with parallel 3-branes, for which the action is (in units with $\kappa = 1$ where κ is 5-D gravitational coupling constant.)

$$S = \int d^x dy \sqrt{| g |} \left(\frac{1}{2} R - \frac{(\nabla \phi^2)}{2} - V(\phi) \right) - \sum_b \int_{y_b} d^4 x \sqrt{| \tilde{g}^b |} \sigma_b(\phi), \tag{1}$$

where the coordinates are (x^μ, y), for $0 \le \mu \le 3$, the bth brane is located at $y = y_b$, g_{ab} is the 5-D metric and $\tilde{g}^b_{\mu\nu}$ is the induced metric on the bth brane. The brane tensions σ_b and the potential V are functions of the bulk scalar field $\phi = \phi(y)$. They seek solutions to the field equations of the form

$$ds^2 = dy^2 + A(y) \left(-dt^2 + e^{2Ht} \delta_{ik} dx^i dx^k \right). \tag{2}$$

To solve the equations of motion, by analogy with the practice in the supergravity theory they define a superpotential $W(\phi)$ by $\dot{\phi} \equiv W'(\phi)$, where an overdot denotes the derivative with respect to y and the prime in W' denotes the derivative with respect to ϕ, and the H is the effective Hubble constant on the our brane. If $H = 0$ then one get

$$V(\phi) = W'^2/8 - W^2/6. \tag{3}$$

Using this equation authors have suggested a simple way for obtaining solutions (see also [2, 3]). On the other hand, the equation (3) get us the supersymmetry connection between

V and W, so one can obtain the Darboux transformation for these equations. This because for the one-dimensional systems [1] the supersymmetry and Darboux transformation are the same [4]. The aim of this work is to use the Darboux transformation to obtain exact solutions of field equations which can be derived by the minimizing the action (1). We are restrict ourselves to studying of the models without cosmological expansion, i.e. when $H = 0$.

One should distinguish between the usual Darboux transformation and Darboux transformations in the brane world. This difference is connected with stick-slip nature of fields in the brane world (the jump conditions at the branes are $\dot\phi(y_b^+) - \dot\phi(y_b^-) = \sigma_b'(\phi_b)$). So one must use some generalization of usual Darboux transformation. As we shall see this generalized transformations can be obtained. Using this transformation we have constructed new exact solutions of brane equations which is some generalization of Rundall-Sundrum solution.

2 Dressing of the Rundall-Sundrum brane.

If $H = 0$ and $u \equiv \dot A/A$ then we get from the (1) and (2) the system ([1]),

$$\dot u = -\frac{2\dot\phi^2}{3} - \frac{2}{3}\sum_b \sigma_b(\phi)\delta(y - y_b), \qquad u^2 = \frac{\dot\phi^2}{3} - \frac{2}{3}V(\phi). \tag{4}$$

The third equation has the form

$$\ddot\phi + 2u\dot\phi = V'(\phi) + \sum_b \sigma'(\phi)\delta(y - y_b),$$

and can be obtained from the system (4), so it is enough to consider one. If we have the single brane which is located at $y = 0$ then $\sigma(\phi(y))\delta(y) = \sigma_0\delta(y)$ where $\sigma_0 = \sigma(\phi(0))$ (we assume that $\phi(0) \neq 0$). Introducing $\psi = \psi(y) = A^2$ we get the Schrödinger equation

$$\ddot\psi = (v(y) - \beta\delta(y) - \lambda)\,\psi, \tag{5}$$

where

$$v(y) = -\frac{8}{3}V(y), \qquad \beta = \frac{4\sigma}{3},$$

and λ is the cosmological constant which can be included into the potential. In this work we are interesting of positive solutions of the (5): $\psi(y;\lambda) > 0$. The solutions of (5) with various values of cosmological term will be necessary for the Darboux transformation (see below).

Let $\psi_1 = \psi(y;\lambda_1)$ and $\psi_2 = \psi(y;\lambda_2)$ are two solutions of the (5). We call ψ_1 the prop function for the DT which has the form [5]:

$$\psi_2 \to \psi_2^{(1)} = \frac{\dot\psi_2\psi_1 - \dot\psi_1\psi_2}{\psi_1}, \qquad v \to v^{(1)} = v - 2\frac{d^2}{dy^2}\log\psi_1, \qquad \lambda_2 \to \lambda_2. \tag{6}$$

DT (6) is the isospectral symmetry of the (5). In other words, the dressed function $\psi_2^{(1)}$ is the solution of dressed equation (5),

$$\ddot\psi_2^{(1)} = \left(v^{(1)} - \beta^{(1)}\delta(y) - \lambda_2\right)\psi_2^{(1)},$$

[1] $\phi = \phi(y)$ so this field is the solution of ODE.

with the new (dressed) potential $v^{(1)}$ and the old eigenvalue λ_2.

The transformation low for the prop function is ([4])

$$\psi_1 \to \psi_1^{(1)} = \frac{1}{\psi_1}, \qquad \ddot{\psi}_1^{(1)} = \left(v^{(1)} - \beta^{(1)}\delta(y) - \lambda_1\right)\psi_1^{(1)}. \tag{7}$$

To use DT (6), (7) we start with the simple initial potential $v = \mu^2$ =const. This is the case of Randall-Sundrum (RS) model for which $V = -3\mu^2/8$. The solution of the (5) with $\lambda = 0$ is

$$\psi(y>0) = \psi_+ = c_1 e^{\mu y} + c_2 e^{-\mu y}, \qquad \psi(y<0) = \psi_- = a_1 e^{\mu y} + a_2 e^{-\mu y}. \tag{8}$$

The jump condition at the brane are

$$\psi_+(0) = \psi_-(0) = \psi(0), \qquad \dot{\psi}_+(0) - \dot{\psi}_-(0) = -\beta\psi(0), \tag{9}$$

so

$$a_1 = c_1 + \frac{\beta(c_1+c_2)}{2\mu}, \qquad a_2 = c_2 - \frac{\beta(c_1+c_2)}{2\mu}.$$

We choose (8) as the prop function. After DT (6) we get

$$\mu^2 - \beta\delta(y) \to \mu^2 + v_\pm^{(1)} - \beta^{(1)}\delta(y),$$

where

$$v_+^{(1)} = -\frac{8\mu^2 c_1 c_2}{\left(c_1 e^{\mu y} + c_2 e^{-\mu y}\right)^2}, \qquad v_-^{(1)} = \frac{8\mu^2(-2c_2\mu + \beta(c_1+c_2))(2c_1\mu + \beta(c_1+c_2))}{\left[(2c_1\mu + \beta(c_1+c_2))\, e^{\mu y} + (2c_2\mu - \beta(c_1+c_2))\, e^{-\mu y}\right]^2}. \tag{10}$$

In the case

$$c_1 > 0, \qquad c_2 > 0, \qquad \mu > \frac{\beta(c_1+c_2)}{2c_2},$$

the solution (10) will be nonsingular.

If we choose the tension as

$$\beta = \frac{2\mu(c_2 - c_1)}{c_1 + c_2}, \tag{11}$$

then $v_-^{(1)}$ can be obtained from the $v_+^{(1)}$ by permutation of c_1 and c_2.

Now we need to find new tension $\beta^{(1)}$. To do this we use (7) and (9). We get $\psi_\pm^{(1)} = 1/\psi_\pm$ therefore at $y \to 0$:

$$\dot{\psi}_+^{(1)}(0) - \dot{\psi}_-^{(1)}(0) = \frac{\dot{\psi}_(0) - \dot{\psi}_+(0)}{\psi^2(0)} = \frac{\beta}{\psi(0)} = -\beta^{(1)}\psi^{(1)}(0),$$

thus the tension changes sign:

$$\beta^{(1)} = -\beta. \tag{12}$$

The metric has the form:

$$A_+(y>0) = \frac{1}{\sqrt{c_1 e^{\mu y} + c_2 e^{-\mu y}}}, \qquad A_-(y>0) = \frac{1}{\sqrt{a_1 e^{\mu y} + a_2 e^{-\mu y}}},$$

so $A(y) \to e^{-\mu y/2}/\sqrt{c_1}$ as $y \to +\infty$ and $A \to e^{\mu y/2}/\sqrt{a_2}$ as $y \to -\infty$. In the case of tweaking (11) one get

$$A(y) \to \frac{1}{\sqrt{c_1 = a_2}} e^{-\mu|y|/2}, \qquad y \to \pm\infty. \tag{13}$$

In this case one can choose $c_2 = a_1 = 0$ to transform (13) into exact RS solution of the system (4). We get initial RS potential with $\beta \to -\beta$ (see (12)).

Turning back to the case of general position we get

$$\dot{\phi}_+^2 = \frac{3\mu^2 c_1 c_2}{(c_1 e^{\mu y} + c_2 e^{-\mu y})^2}, \qquad y > 0,$$

and after simple calculations one have the well-known sine-Gordon potential $V_+ = V(\phi_+)$,

$$V_+ = \frac{3\mu^2}{4} \cos\left(\frac{4\phi_+}{\sqrt{3}}\right), \tag{14}$$

with

$$\phi_+(y) = \pm\frac{\sqrt{3}}{2} \arctan\left[\sinh\left(\mu y + \frac{1}{2}\log\frac{c_1}{c_2}\right)\right]. \tag{15}$$

If $y < 0$ then we get the same sine-Gordon potential (14) with $\phi_+ \to \phi_-$, where the solution ϕ_- can be obtained from the (15) by the substitution $c_{1,2} \to a_{1,2}$.

3 n-times Darboux transformation.

A single act of dressing (6) can be iterated n times [6]. Let consider the two-times DT. The solution of the (5) with $v = \mu^2$ and $\lambda = \mu^2 - \nu^2$ is

$$\phi(y > 0) = \phi_+ = b_1 e^{\nu y} + b_2 e^{-\nu y}, \qquad \phi(y < 0) = \phi_- = d_1 e^{\nu y} + d_2 e^{-\nu y}.$$

The jump condition (9) for the dressed functions will be valid if

$$d_1 = \frac{(b_1 + b_2)\beta + 2b_1\nu}{2\nu}, \qquad d_2 = \frac{2b_2\nu - (b_1 + b_2)\beta}{2\nu}.$$

The first remark of material significance: in this case the jump condition is not valid for the $\phi_\pm$ itself therefore $\phi_\pm$ are not solution of initial Schrödinger equation. So we can consider these transformations as some generalization of usual Darboux transformations.

At last one get

$$\beta^{(2)} = \beta,$$

and dressed potential which has bulky form. Using tweaking (11) with $c_2 = 0$ one get

$$v_\pm^{(2)}(y) = v_\pm^{(1)}(y) - 2\frac{d^2}{dy^2}\log\phi_\pm^{(1)}(y) - \mu^2 + \nu^2,$$

where $\phi_\pm^{(1)}(y)$ can be calculated by the (6). At last we get (for the $y > 0$)

$$v_+^{(2)} = \nu^2\left(1 - \frac{8b_1 b_2(\mu^2 - \nu^2)}{(b_1(\mu - \nu)e^{\nu y} + b_2(\mu + \nu)e^{-\nu y})^2}\right).$$

In the case of general position we can use n-times DT to obtain solutions with RS asymptotic form . To do it we use the Crum formulas [6] starting out from the set of exact solutions of the equations (5) with $v = \mu^2$. As a result one get (omitting indices)

$$A_\pm^{(n)}(y) = \sqrt{\frac{\Delta^{(\pm)}(y)}{D^{(\pm)}(y)}}, \qquad v_\pm^{(n)}(y) = v - 2\frac{d^2}{dy^2}\log\Delta^{(\pm)}(y), \tag{16}$$

where

$$\Delta^{(\pm)}(y) = \begin{vmatrix} \psi^{[n-2]}_{n-1,\pm} & \psi^{[n-3]}_{n-1,\pm} & \dots & \psi_{n-1,\pm} \\ \psi^{[n-2]}_{n-2,\pm} & \psi^{[n-3]}_{n-2,\pm} & \dots & \psi_{n-2,\pm} \\ . & & & \\ . & & & \\ \psi^{[n-2]}_{1,\pm} & \psi^{[n-3]}_{1,\pm} & \dots & \psi_{1,\pm} \end{vmatrix}, \qquad D^{(\pm)}(y) = \begin{vmatrix} \psi^{[n-1]}_{n,\pm} & \psi^{[n-2]}_{n,\pm} & \dots & \psi_{n,\pm} \\ \psi^{[n-1]}_{n-1,\pm} & \psi^{[n-2]}_{n,-1\pm} & \dots & \psi_{n-1,\pm} \\ . & & & \\ . & & & \\ \psi^{[n-1]}_{1,\pm} & \psi^{[n-2]}_{1,\pm} & \dots & \psi_{1,\pm} \end{vmatrix},$$

and

$$\psi^{[m]}_{k,\pm} = \mu_k^m \left(c^{(\pm)}_{1,k} e^{\mu_k y} + (-1)^m c^{(\pm)}_{2,k} e^{-\mu_k y} \right), \qquad k = 1, .., n.$$

The jump conditions take the form

$$G(0) = \begin{vmatrix} \Delta^{(+)}(0) & \Delta^{(-)}(0) \\ D^{(+)}(0) & D^{(-)}(0) \end{vmatrix} = 0,$$
$$\frac{D^{(+)}(0) D^{(-)}(0)}{D^{(-)}(0)\Delta^{(+)}(0) + D^{(+)}(0)\Delta^{(-)}(0)} \left(\frac{d}{dy} \frac{G(y)}{D^{(+)}(y) D^{(-)}(y)} \right)_{y=0} = -\frac{\tilde\beta}{2}. \tag{17}$$

The second remark of material significance: we use here the relaxed jump condition in comparison with one from the preceding Section. We only need jump conditions for the n-dressed solutions $\psi^{(n)}$ and don't need it for the intermediate solutions $\psi^{(k)}$ with $k < n$.

This approach can be easy generalized to the case of K branes. To do it we need replace the term $\beta\delta(y)$ in the (5) by

$$\sum_{b=1} \beta_b \delta(y - y_b),$$

and replace two jump conditions (17) by $2K$ jump conditions at $y = y_b$. If μ_k and $\tilde\beta$ are assigned then we have $4n$ free parameters $c^{(\pm)}_{1,k}$, $c^{(\pm)}_{2,k}$ therefore we can construct $K = 2n$ branes. If μ_k are free parameters too then the calculation of K is more difficult task. The naive rough estimate get $K = [2.5n]$, where square brackets denotes the integer part.

In the case of general position we can't reconstruct the form of potential $V(\phi)$ as function of ϕ. Nevertheless, all these solutions have RS asymptotic, because $V^{(n)}(y) \to -c^2$ (c=const) as $y \to \pm\infty$. Reader can easy obtain a great many exact solutions with RS asymptotic using (16), (17) .

And last but not least, to calculate the n-times dressed Ricci scalar one can use the formula

$$R^{(n)}_\pm = -\frac{4\ddot A^{(n)}_\pm A^{(n)}_\pm + (\dot A^{(n)}_\pm)^2}{(A^{(n)}_\pm)^2}.$$

4 Shape-invariant potentials.

Another way to obtain exact soluble potentials via DT is the use of shape invariants [7]. If an initial potential is function of y and free parameters a_i : $v = v(y; a_i)$, and after DT we get $v^{(1)} = v^{(1)}(y; a^{(1)}_i)$ then v the shape-invariant (SP) potential.

There are many SP potentials in the quantum mechanics. The far-famed example is the harmonic oscillator [4]. The exact soluble potentials from [1] (for the models without cosmological expansion) are SP potentials. We restrict ourselves to considering three examples from this article.

A. Even superpotential.

In the case of DFGK model [8] we get $(A(0) = 1)$

$$\log A(y) = -\frac{ay}{3} + g\left(1 - e^{2by}\right).$$

In this case

$$v(y) = \frac{8}{3}gb(2a - 3b)e^{2by} + 16g^2b^2e^{4by} + \frac{4a^2}{9}.$$

After DT

$$v \rightarrow v^{(1)} = v - 4\frac{d^2}{dy^2}\log A,$$

we get

$$v^{(1)}(y) = \frac{8}{3}gb(2a + 3b)e^{2by} + 16g^2b^2e^{4by} + \frac{4a^2}{9}.$$

Thus $v^{(1)}$ can be obtained from the v by the substitutions: $a \rightarrow -a$ and $g \rightarrow -g$. In other words $v(y)$ is SP potential.

B. Odd superpotential.

In this case we have

$$\log A(y) = -ay - by^2.$$

After calculation we get

$$v(y) = 4\left(2by + a\right)^2 - 4b.$$

It is nothing but the harmonic oscillator and, of course, it is SP potential: $v^{(1)} = v + const$.

C. Exponential potential.

Superpotential is $W(\phi) = 2be^{-\phi}$, where $\phi(y) = \log(a - by)$. Therefore

$$v(y) = \frac{c}{(by - a)^2}.$$

It is well known SP potential [9]: $v^{(1)} = const \times v$.

5 Conclusion.

DT gives a link between solvable problems and one finds that most (if not all) exactly solvable potentials, starting out from the harmonic oscillator and ending off the finite-gap potentials [10], can be obtained by these transformations. The physical sense of these potentials is not clear to me but regard must be paid to the fact that we want only to demonstrate and advertise DT as power tool to construction of exact soluble potentials in 5-D gravity with bulk scalar field in models without cosmological expansion. There are two imperfections of this method:
1. DT is working if the Hubble root $H = 0$. It is not our present universe.

2. DT is working only for the 5-D gravity. If $D > 5$ then the warp factor is multivariable function. The bitter truth is that we have not articulate theory of DT for many-dimensional equations [11, 12].

But if $H = 0$ and $D = 5$ the DT (with the first remark from the Sec. 3) is the best way to study exact solvable potentials.

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